Bernoulli Wavelet Collocation Method for the Solution of Delay Differential Equations

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Abstract:
Bernoulli wavelet collocation technique for solving linear and nonlinear delay differential equations (DDEs) is presented in this article. As the number of Bernoulli wavelet bases increases, the resulting solution becomes more optimal. Tables and figures confirm the method's stability.

Keywords: With the collocation approach, Bernoulli wavelets, and linear and nonlinear delay differential equations solved.
MSC: 33E99, 34K06, 34K07

1. Introduction:

The system in which the rate of change of a function relies on its values at previous time is known as a delay differential equation (DDE)[1-4]. Because they can represent memory in this system, these equations show to be more realistic than regular differential equations. Both algebraic and transcendental functions are used in DDEs, however the characteristic equation of a DDE including a transcendental function is notoriously difficult to analyse using ordinary differential equations. We have used Bernoulli wavelets to analyse DDEs.

Consider general form of DDEs [6-7],

$$\frac{d^2}{dt^2} y(t) = f\left(t, y(t), \frac{d}{dt} y(t)\right) + g\left(y(t), \frac{d}{dt} y(t), y\left(\frac{t}{a}\right), y(t-b), \frac{d}{dt} y\left(\frac{t}{a}\right), \ldots\ldots\right) + h(t). \tag{1}$$

2. Bernoulli Wavelets

Bernoulli polynomials

Bernoulli polynomials are defined in general as [9-10]:

$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-i} t^i, \quad n = 0,1,2,..., \tag{2}$$

where, α_i, $i = 0,...,n$ are Bernoulli numbers. For instance, starting five Bernoulli numbers are:

$$\alpha_0 = 1,\ \alpha_1 = -\frac{1}{2},\ \alpha_2 = \frac{1}{6},\ \alpha_3 = 0,\ \alpha_4 = -\frac{1}{30}.$$

And the first four Bernoulli polynomials are:

$B_0(t)=1$, $B_1(t)=t-\frac{1}{2}$, $B_2(t)=t^2-t+\frac{1}{6}$, $B_3(t)=t^3-\frac{3}{2}t^3+\frac{1}{2}t$.

Bernoulli wavelets

Bernoulli wavelets are defined as follows [11]:

$$\psi_{j,n}(t)=\begin{cases}2^{\frac{l-1}{2}}\bar{B}_n(2^{l-1}t-j+1), & \frac{j-1}{2^{l-1}}\le x<\frac{j}{2^{l-1}},\\ 0, & otherwise,\end{cases} \tag{3}$$

in which

$$\underline{B}_n(t)=\begin{cases}1, & n=0,\\ \frac{1}{\sqrt{(((-1)^{n-1}(n!)^2)/((2n)!))\alpha_{2n}}}B_n(t), & n>0,\end{cases}$$

where, $n=0,1,...,M-1$ denotes the order of the Bernoulli polynomials and $j=1,2,...,2^{l-1}, l\in N$.

Approximation of function

Suppose $f(t)\in L^2[0,1)$ is expanded in terms of the Bernoulli wavelets as

$$f(t)=\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}c_{jn}\psi_{jn}(t). \tag{4}$$

Truncating the above infinite series, we get

$$f(t)\approx\sum_{j=1}^{2^{l-1}}\sum_{n=0}^{M-1}c_{jn}\psi_{jn}(t)=C^T\psi(t)=f^n(t), \tag{5}$$

where, C and $\psi(x)$ are $n\times1$ ($n=2^{l-1}M$) matrices given by

$$C=\left[c_{10},c_{11},...,c_{1,M-1},c_{20},...,c_{2,M-1},...,c_{2^{l-1},0},...,c_{2^{l-1},M-1}\right]^T, \tag{6}$$

$$\psi(t)=\left[\psi_{10}(t),\psi_{11}(t),...,\psi_{1,M-1}(t),\psi_{20}(t),...,\psi_{2,M-1}(t),...,\psi_{2^{l-1},0}(t),...,\psi_{2^{l-1},M-1}(t)\right]^T \tag{7}$$

3. Method of solution for linear and nonlinear delay differential equations

Consider nonlinear delay differential equation

$$\frac{d^2}{dt^2}y(t)=f\left(t,y(t),\frac{d}{dt}y(t)\right)+g\left(y(t),\frac{d}{dt}y(t),y\left(\frac{t}{a}\right),y(t-b),\frac{d}{dt}y\left(\frac{t}{a}\right),......\right)+h(t). \tag{8}$$

In the above equation $f\left(t,y(t),\frac{d}{dt}y(t)\right)$ consists of linear terms only $h(t)$, consists of nonhomogeneous part and $g\left(y(t),\frac{d}{dt}y(t),y\left(\frac{t}{a}\right),y(t-b),\frac{d}{dt}y\left(\frac{t}{a}\right).....\right)$ consists of nonlinear and delay part only.

Let us consider a function $\phi(t)$, which may be any function and satisfies the given conditions.

No we will replace $y(t)$ by $\phi(t)$ in nonlinear part and delay part and general equation of delay differential equation then we have,

$$\frac{d^2}{dt^2} y(t_i) = f\left(t_i,\ y(t_i),\ \frac{d}{dt} y(t_i)\right) + g\left(\phi(t_i),\ \frac{d}{dt}\phi(t_i), \phi\left(\frac{t_i}{a}\right), \phi(t_i - b),\ \frac{d}{dt}\phi\left(\frac{t_i}{a}\right), ...\right) + h(t_i). \quad (9)$$

Now we have to use the Bernoulli wavelet bases as $y(t)$ then we get series solution on solving above equation with aid of collocation points as:

$$\{t_i\} = \{\frac{1}{2}(1 + \cos\frac{(i-1)\pi}{2^{l-1}M-1})\} \quad i = 2.3, ... \quad (10)$$

Then we get system of algebraic equation with Bernoulli coefficients. On solving these equations we get Bernoulli wavelet solution for linear and nonlinear delay differential equations of fractional order.

4. Bernoulli wavelet collocation method implementation

Example1: Nonlinear Pantograph equation of [7],

$$\frac{d^2}{dt^2} y(t) + y(t) - 5\left(y\left(\frac{t}{2}\right)\right)^2 = 0, \quad (11)$$

with boundary conditions,

$y(0) = 1, \quad y(1) = 1.3534e\text{-}01.$

Exact solution is for $\alpha = 2 \quad y(t) = e^{-2t}$. (12)

On applying Bernoulli wavelet method for above example then we get,

$$\frac{d^2}{dt^2} y(t) + y(t) - 5\left(\phi\left(\frac{t}{2}\right)\right)^2, \quad (13)$$

consider $\phi(t) = e^{-2t}$, then equation(13) becomes,

$$\frac{d^2}{dt^2} y(t_i) + y(t_i) - 5(e^{-t_i})^2, \quad (14)$$

Replacing $y(t_i)$ by Bernoulli wavelets then equation (14) becomes,

$$\frac{d^2}{dt^2}\left(\begin{array}{l} a1(1.4142) + a2(2\sqrt{6}(2 * t_i - 1/2)) + a3(6\sqrt{2}\sqrt{5}(4 * t_i^2 - 2t_i + 1/6)) \\ + a4(4\sqrt{105}(8t_i^3 - 6t_i^2 + t_i)) + a5(10\sqrt{2}\sqrt{21}(16t_i^4 - 16t_i^3 + 4t_i^2 - 1/30)) \end{array}\right)$$
$$+\left(\begin{array}{l} a1(1.4142) + a2(2\sqrt{6}(2 * t_i - 1/2)) + a3(6\sqrt{2}\sqrt{5}(4 * t_i^2 - 2t_i + 1/6)) \\ + a4(4\sqrt{105}(8t_i^3 - 6t_i^2 + t_i)) + a5(10\sqrt{2}\sqrt{21}(16t_i^4 - 16t_i^3 + 4t_i^2 - 1/30)) \end{array}\right) - 5(e^{-t_i})^2 = 0, \quad (15)$$

$1 - 0.500000015x^2 + 1.51930931\,10^{-12}x^3 + 0.04166663x^4 + 3.657918\,10^{-10}x^5 - 0.00138891x^6 + 7.4489920\,10^{-9}x^7 + 0.00002478x^8 + 2.318362\,10^{-8}x^9 - 2.9565591\,10^{-7}x^{10} + 9.81057583\,10^{-9}x^{11}$

on solving above equation with aid of MATLAB software we get coefficients for k=2, M=7, then we have,

a1= 4.4674e-01
a2= -1.2906e-01
a3= 1.6641e-02
a4= -2.4034e-03
a5= 2.4647e-04

On substituting these coefficients in Eqn. (15) we get Bernoulli wavelet series solution as,

$$y(t) = 0.255569x^4 - 1.04365x^3 + 1.91792x^2 - 1.99452x + 1 \quad (16)$$

Table 1: Bernoulli wavelet solution for k=2 and M=5 and M=7 for example 1.

y	**Exact solution** α =2	**BWS at** k=2; M=3	**BWS at** k=2; M=5	**AE at** k=2; M=7
0.1	8.1873e-01	8.4813e-01	8.1871e-01	8.1873e-01
0.2	6.7032e-01	7.1080e-01	6.6987e-01	6.7032e-01
0.3	5.4881e-01	5.8800e-01	5.4815e-01	5.4881e-01
0.4	4.4933e-01	4.7974e-01	4.4881e-01	4.4933e-01
0.5	3.6788e-01	3.8601e-01	3.6774e-01	3.6788e-01
0.6	3.0119e-01	3.0681e-01	3.0144e-01	3.0119e-01
0.7	2.4660e-01	2.4215e-01	2.4701e-01	2.4660e-01
0.8	2.0190e-01	1.9202e-01	2.0219e-01	2.0190e-01
0.9	1.6530e-01	1.5643e-01	1.6531e-01	1.6531e-01
1.0	1.3534e-01	1.3537e-01	1.3532e-01	1.3535e-01

Table 1: Absolute error (AE) between Bernoulli wavelet solution and exact solution for k=2 and M=5 and M=7 for example 1.

y	**BWS at** k=2; M=3	**BWS at** k=2; M=5	**AE at** k=2; M=7
0.1	2.9400e-02	1.9428e-05	5.1355e-06
0.2	4.0478e-02	4.4506e-04	4.1236e-06
0.3	3.9189e-02	6.6063e-04	2.8037e-07
0.4	3.0408e-02	5.1805e-04	2.4915e-06

0.5	1.8128e-02	1.3985e-04	1.4385e-06
0.6	5.6169e-03	2.4106e-04	5.6634e-07
0.7	4.4478e-03	4.1262e-04	3.1329e-06
0.8	9.8752e-03	2.9098e-04	8.0654e-06
0.9	8.8715e-03	8.4837e-06	1.0686e-05
1.0	3.2095e-05	1.4368e-05	1.1837e-05

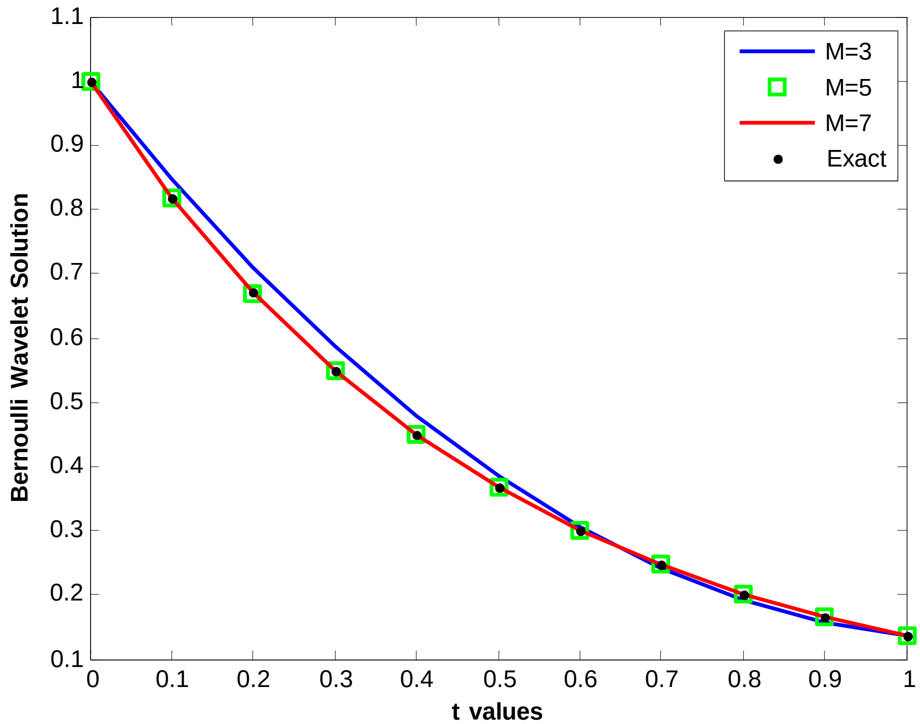

Fig. 1: Comparison of Bernoulli wavelet solution for different fractional values of M at k=2 for example 1.

Example 2: Bernoulli wavelet collocation method for nonlinear delay differential equation[6],

$$\frac{d^2}{dt^2}y(t)-\frac{8}{3}\frac{d}{dt}\left(y\left(\frac{t}{2}\right)\right)y(t)+8t^2y\left(\frac{t}{2}\right)+\frac{4}{3}+\frac{22}{3}t+7t^3+\frac{5}{3}t^3=0, \quad (17)$$

with boundary conditions,

$y(0)=1,\ y(1)=1.$

Exact solution of Eqn. (17) is $y(t)=1+t-t^3.$ (18)

Table 2: Bernoulli wavelet solution for different values of M and k=2 for example 2.

t	**Exact solution**	**BWS at** k=2; M=3	**BWS at** k=2; M=5	**AE at** k=2; M=7
0.1	1.0990e+00	1.1350e+00	1.0990e+00	1.0990e+00
0.2	1.1920e+00	1.2400e+00	1.1920e+00	1.1920e+00
0.3	1.2730e+00	1.3150e+00	1.2730e+00	1.2730e+00
0.4	1.3360e+00	1.3600e+00	1.3360e+00	1.3360e+00
0.5	1.3750e+00	1.3750e+00	1.3750e+00	1.3750e+00
0.6	1.3840e+00	1.3600e+00	1.3840e+00	1.3840e+00
0.7	1.3570e+00	1.3150e+00	1.3570e+00	1.3570e+00
0.8	1.2880e+00	1.2400e+00	1.2880e+00	1.2880e+00
0.9	1.1710e+00	1.1350e+00	1.1710e+00	1.1710e+00
1.0	1.0000e+00	1.0000e+00	1.0000e+00	1.0000e+00

t	**BWS at** k=2; M=3	**BWS at** k=2; M=5	**AE at** k=2; M=7
0.1	3.5997e-02	7.2587e-06	7.2587e-06
0.2	4.7997e-02	6.8370e-06	6.8371e-06
0.3	4.1997e-02	6.3764e-06	6.3764e-06
0.4	2.3998e-02	5.9008e-06	5.9008e-06
0.5	9.2633e-07	5.4346e-06	5.4346e-06
0.6	2.3999e-02	5.0022e-06	5.0023e-06
0.7	4.1998e-02	4.6282e-06	4.6284e-06
0.8	4.7995e-02	4.3373e-06	4.3380e-06
0.9	3.5993e-02	4.1543e-06	4.1558e-06
1.0	1.0188e-05	4.1042e-06	4.1071e-06

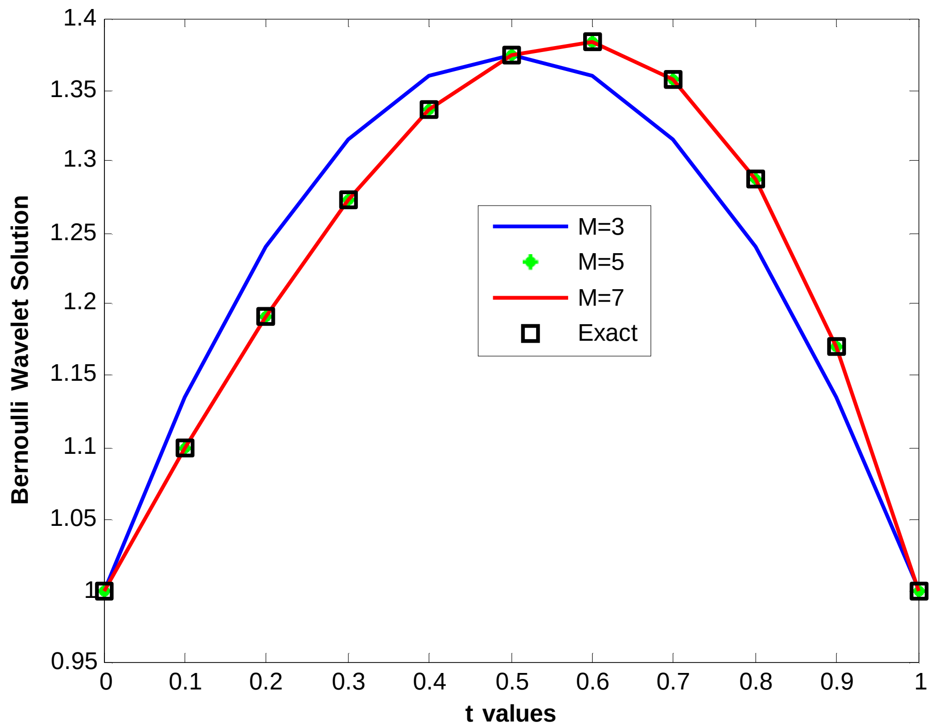

Fig. 2: Comparison of Bernoulli wavelet solution for different fractional values of M and at k=2 for example 2.

Example 3: Bernoulli wavelet steps method for nonlinear Delay differential equation [8],

$$\frac{d^2}{dt^2}y(t) - 1 + 2y^2\left(\frac{t}{2}\right) = 0, \quad (19)$$

With boundary conditions,

$y(0) = 1, \quad y(1) =$ 5.4030e-01.

Exact solution of above problem is $y(t) = \cos t$ (20)

Table 3: Bernoulli wavelet solution and AE between exact solution and BWCM solution for k=2, M=6 and M=8 for example 3.

t	**Exact solution**	**BWS at** k=2; M=6	**BWS at** k=2; M=8	**AE at** k=2; M=6	**AE at** k=2; M=8
0.1	9.9500e-01	9.9500e-01	9.9500e-01	6.1476e-07	0
0.2	9.8007e-01	9.8007e-01	9.8007e-01	1.5326e-06	3.9052e-10
0.3	9.5534e-01	9.5534e-01	9.5534e-01	8.4663e-07	7.5078e-10
0.4	9.2106e-01	9.2106e-01	9.2106e-01	5.5072e-07	1.0834e-10
0.5	8.7758e-01	8.7758e-01	8.7758e-01	1.2479e-06	1.3913e-10

0.6	8.2534e-01	8.2533e-01	8.2534e-01	6.4197e-07	1.6827e-10
0.7	7.6484e-01	7.6484e-01	7.6484e-01	6.5230e-07	1.9643e-10
0.8	6.9671e-01	6.9671e-01	6.9671e-01	1.2692e-06	2.2388e-10
0.9	6.2161e-01	6.2161e-01	6.2161e-01	3.8163e-07	2.5074e-10
1.0	5.4030e-01	5.4030e-01	5.4030e-01	2.4597e-07	2.7765e-10

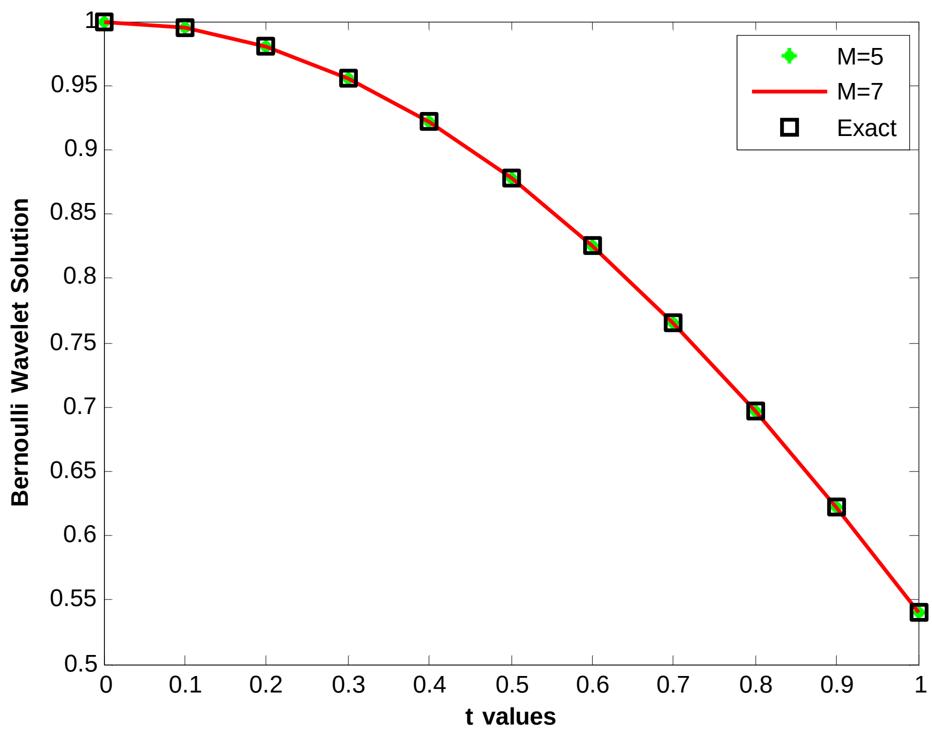

Fig. 3: Comparison of Bernoulli wavelet solution for different fractional values of M and at k=2 for example 3.

5. Conclusion:

For the purpose of solving linear and nonlinear delay differential equations using collocation points, the Bernoulli wavelet collocation technique has been examined in this paper. We are finding solutions that are quite close to the precise answer for several Bernoulli wavelet bases. When the value of M increases, the absolute inaccuracy likewise increases. We used tables and numbers to check for correctness and practicality.

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